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Foucault Pendulum Experiment by Kamerlingh Onnes and Degenerate Perturbation Theory

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(Received 15 July 1969)

The paper pays homage to H. Kamerlingh Onnes by pointing out that he constructed the first properly functioning model of the Foucault pendulum. His success was largely due to an analysis which takes into account the unavoidable mechanical asymmetry of the pendulum. Based on the analysis he was able to eliminate the asymmetry and to obtain the desired performance which earlier workers had realized only to a limited extent. Onnes' analysis is reproduced with more modern tools such as vector notation and, especially, the concept of eigenfunctions. His analysis is probably one of the earliest examples of a particular type of doubly degenerate perturbation theory. Two different effects compete for lifting the degeneracy, one favoring "circular polarization," the other "linear polarization." The resulting eigenfunctions represent "elliptical polarization." The tutorial significance of this perturbation theory is emphasized by mentioning other fields where it is applied if perhaps disguised by a different jargon and analytic technique. Examples include electromagnetic theory (microwave cavities for circular polarization), laser optics (mode analysis of ring lasers for rotation sensing), and quantum theory (quenching of orbital angular momentum by crystalline field).

INTRODUCTION

The liquefaction of helium and the discovery of superconductivity by H. Kamerlingh Onnes are events of such far-reaching consequences to the evolution of physics that his name is deservedly familiar to physicists and interested lay people everywhere. Few people realize, however, that his first work constitutes a significant contribution to physics, too. This is his 1879 Groningen dissertation[1] entitled "Nieuwe bewijzen voor de aswenteling der aarde."[2] The experimental proof for the axial rotation of the earth consisted in the demonstration of a properly functioning Foucault pendulum. As predicted by the unsophisticated theory of the device,[3–5] he found that the plane containing the pendulum excursion rotates gradually about the vertical. In 24 h the rotation amounts to $2\pi \sin\lambda$, where λ is the geographical latitude of the laboratory.

The same experiment was carried out earlier first by Foucault and then by Bravais, however, with somewhat limited success. Initially the pendulum was excited to a motion whose horizontal projection was a straight line. For some time thereafter, usually not more than one hour,

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the projection of the pendulum motion remained essentially linear and changed in direction as expected. Observation for longer periods of time showed, however, that the motion gradually "deteriorated" into one with an elliptical projection. No explanation was available at the time,[6] and, as a result, nobody was able to construct a pendulum which would give the expected performance for longer periods of time.

The subject of the dissertation had been suggested to Kamerlingh Onnes by his former teacher Gustav Kirchhoff, then at Berlin. Undoubtedly the unsatisfactory results of the earlier Foucault pendulum experiments were then brought to Onnes' attention. It is not clear, on the other hand, whether Kirchhoff had been able to suggest to Onnes the likely theoretical cause for the occurrence of elliptical orbits.

The first section of the present paper will briefly sketch the theoretical analysis given in Kamerlingh Onnes' dissertation. Nowadays this can be done in a considerably more compact and transparent form with the help of mathematical and conceptual tools which were not generally accepted then, such as vector algebra, complex notation, and especially eigenfunctions and eigenvalues. Most likely this analysis is historically the first example of degenerate perturbation theory as it is now called. In the idealized, unperturbed situation the two modes of oscillation (or eigenfunctions) of the pendulum are degenerate. One type of interaction which tends to lift the degeneracy is the rotation of the earth. The corresponding eigenfunctions are circular orbits of the pendulum. Another interaction which may split the degeneracy is an (unintentional) difference between the principal moments of inertia of the pendulum. Here the resulting eigenfunctions are linear pendulum oscillations. In the general case, both interactions compete and lead to a pair of orthogonal elliptical orbits as eigenfunctions. The theoretical insight permitted Onnes to identify the eigenfunctions by observations. Once these were known he was able to adjust the mass distribution on the pendulum so as to make equal the two principal moments of inertia. After that is accomplished, the eigenfunctions are circular orbits and the Foucault pendulum performs as expected from the unsophisticated theory.

The first section of the paper is illustrated with some original figures of the dissertation. They pertain both to the theoretical analysis and the experimental technique used. The design drawings of parts of the equipment may serve as an early example of Kamerlingh Onnes' mastery of experimental techniques. Nevertheless, in the context of the present paper, the emphasis is on the tutorial significance of the theoretical analysis. It is felt that Onnes' treatment of the Foucault pendulum is a very lucid example of the use of eigenfunctions and degenerate perturbation theory. The example recommends itself for introducing students to these concepts at an early stage in their career.

The second section shows an application of the same type of degenerate perturbation calculation within the framework of electromagnetic theory. Microwave cavities for circular polarization are considered. An "unperturbed" situation is assumed in which two cavity modes are degenerate. Unintentional asymmetries in the cavity design lift the degeneracy and lead to eigenmodes with linear polarization. The insertion of gyromagnetic material such as magnetized ferrite also lifts the degeneracy, but leads to circularly polarized eigenmodes. In general, both effects compete and the two resultant eigenmodes are elliptical. The theoretical treatment is an adaptation to the degenerate case of the well-known (but never published) Bethe–Schwinger cavity perturbation theory.[7,8]

The third section discusses a similar problem in the field of laser optics. It is known that ring lasers enable one to measure the rotation of the laser frame relative to inertial space.[9–12] In this device the eigenmodes are electromagnetic waves around the closed optical path. It suffices to consider a single transverse and longitudinal mode. Then, in the "unperturbed" state, there are two degenerate eigenfunctions which may be represented either as oppositely running waves, or as standing waves displaced by a quarter wavelength, or by linear superpositions of these. Viewed in the frame associated with the laser, the degeneracy is lifted by rotation. The appropriate eigenfunctions are then running waves. Dielectric or lossy elements in the optical path also lift the degeneracy, but they lead to standing wave eigenfunctions. In general, both effects must be considered so that the eigenfunctions will be intermediate between running and standing waves.

The results of this linear perturbation calculation indicate that the ring laser cannot identify low rotation rates. Experimentally one does indeed find a "locking" phenomenon[12] which prevents the measurement of low rotation rates. The correct explanation of locking, however, involves nonlinear interactions between the laser medium and the electromagnetic modes[13] which are usually much stronger than the linear perturbations. Thus it would seem difficult to confirm the results of the linear perturbation calculation.

In a fourth section it is shown that the same degenerate perturbation theory is used in quantum theory. The example discussed involves the "quenching of angular momentum" by crystalline fields. When acting alone, these fields lead to the selection of electronic wave functions with vanishing angular momentum. Other interactions, such as spin–orbit coupling and Zeeman effect, by themselves would select wave functions having maximum angular momentum. A discussion of the actual general problem would involve much detail, which is unessential for the present tutorial purposes. Therefore a somewhat truncated problem is considered in the text. It is seen that the combined interactions lead to wave functions with an angular momentum which is intermediate between zero and the maximum value.

DEGENERATE PERTURBATION THEORY OF THE FOUCAULT PENDULUM

The pendulum designed by Kamerlingh Onnes made use of a rigid pendulum body. Thus the equation of motion should be based on the torque acting on the moment of inertia. However, the significant facts may just as well be discussed in terms of the conventional simple pendulum model where the equation of motion describes the forces which act on the pendulum mass m,

$$m\ddot{\bar{r}}+2m\bar{\Omega}\times\dot{\bar{r}}+\nabla U=0. \tag{1}$$

Here $\bar{r}$ indicates the coordinates of m in the terrestrial frame, a dot $\dot{}$ denotes the time derivative, $\bar{\Omega}$ is the angular rotation vector of the earth, and U is the gravitational potential. The second term is known as the Coriolis force. A centrifugal term, being second order in Ω, need not be considered.

It is not likely that the potential U has perfect rotational symmetry about the vertical. This can be due to the design of the suspension which in effect may lead to two different pendulum lengths for oscillation in two orthogonal directions or, as mentioned before, it may be due to a difference between the principal moments of inertia. Now introduce an x, y coordinate system in the horizontal plane. Identify the x and y directions with those where the second derivative of U is maximal and minimal, respectively. For small excursions in the x and y directions, the excursion in the vertical z direction is small of second order and may be neglected. Thus Eq. (1) becomes

$$m\ddot{x}-2m\Omega_z\dot{y}+mgx/l_x=0$$

$$m\ddot{y}+2m\Omega_z\dot{x}+mgy/l_y=0. \tag{2}$$

Here $\Omega_z=\Omega\sin\lambda$, with λ the geographical latitude, and l_x and l_y are the pendulum lengths which are effective for oscillations in the x and y directions, respectively. Now let

$$(g/l_x)^{1/2}=\omega+\delta$$

$$(g/l_y)^{1/2}=\omega-\delta$$

$$x=a\cos[(\omega+\Delta)t+\varphi]$$

$$y=b\sin[(\omega+\Delta)t+\varphi]$$

with Ω_z, δ, $\Delta\ll\omega$. Then Eq. (2) reduces to

$$(\Delta-\delta)a+\Omega_z b=0$$

$$\Omega_z a+(\Delta+\delta)b=0. \tag{3}$$

The homogeneous Eqs. (3) permit a nontrivial solution only if the determinant vanishes. The resulting equation

$$\Delta^2=\delta^2+\Omega_z^2 \tag{4}$$

has two solutions $\Delta=\pm\sqrt{\delta^2+\Omega_z^2}$

$$\Delta_1=+A,\qquad \Delta_2=-A,\qquad A=(\delta^2+\Omega_z^2)^{1/2}. \tag{5}$$

It is seen that the splitting of the degeneracy depends in part on the earth's rotation and in part on the asymmetry of the pendulum construction.

The eigenfunctions corresponding to the roots Δ_1 and Δ_2 are elliptical motions of the pendulum bob. Introducing

$$\cos\alpha=\Omega_z/[2A(A-\delta)]^{1/2}=(A+\delta)^{1/2}/(2A)^{1/2}$$

$$\sin\alpha=(A-\delta)^{1/2}/(2A)^{1/2}=\Omega_z/[2A(A+\delta)]^{1/2}, \tag{6}$$

the eigenfunctions in normalized form $(x^2+y^2=1)$ are

$$\psi_1(t)=x_1(t)=-\cos\alpha\,\cos[(\omega+A)t+\varphi_1]$$
$$=y_1(t)=\sin\alpha\,\sin[(\omega+A)t+\varphi_1], \qquad (7a)$$

and

$$\psi_2(t)=x_2(t)=\sin\alpha\,\cos[(\omega-A)t+\varphi_2]$$
$$=y_2(t)=\cos\alpha\,\sin[(\omega-A)t+\varphi_2]. \qquad (7b)$$

The first and second eigenfunctions describe ellipses with interchanged major and minor axes and with opposite sense of circulation. They are orthogonal in the sense that $x_1x_2+y_1y_2$ vanishes when averaged over a suitable time interval. If the pendulum is excited to an orbit described by an eigenfunction, it will continue to move in this orbit without change (except perhaps for damping which is unavoidable in practice). Any other excitation involves both eigenfunctions. Due to the frequency difference A between them, the pattern of the pendulum motion changes with time. Particularly interesting eigenfunctions are obtained in two limiting cases. In the "ideal Foucault situation" one has $\delta=0$ and hence $\Delta_1=-\Delta_2=\Omega_z$. The eigenfunctions are counter-rotating circular orbits. The opposite extreme is found at the earth's equator where $\Omega_z=0$. The eigenfunctions are linear oscillations in the x and y directions, respectively.

The Foucault experiment amounts to the simultaneous excitation of two eigenfunctions and the observation of what is essentially a beat phenomenon between them. The effect expected by Foucault is observed only, however, if the pendulum is mechanically symmetric, i.e., $\delta=0$. Now small asymmetries are difficult to avoid. The approach chosen by Kamerlingh Onnes was therefore to determine the asymmetry by observing the evolution of the pendulum orbits. This then enabled him to make corrections in the mass distribution of the pendulum such that the asymmetry was systematically eliminated.

For an illustration of this procedure, consider a general motion of the pendulum. It may be represented as the superposition of ψ_1 and ψ_2 from Eq. (7) with constant real coefficients, c_1 and c_2, and with appropriate phases, φ_1 and φ_2. These four adjustable parameters are determined by the initial conditions. Typically the pendulum starts from rest at a position of finite excursion, for example, on the unit circle. Thus

$$x(0)=\cos\vartheta, \qquad y(0)=\sin\vartheta, \qquad \dot{x}(0)=\dot{y}(0)=0. \qquad (8)$$

The superposition of eigenfunctions which satisfies Eq. (8) is

$$\psi(t)=x(t)=\cos\vartheta(\cos\omega t\,\cos At-\cos2\alpha\,\sin\omega t\,\sin At)$$
$$+\sin\vartheta\,\sin2\alpha\,\cos\omega t\,\sin At$$
$$=y(t)=\sin\vartheta(\cos\omega t\,\cos At+\cos2\alpha\,\sin\omega t\,\sin At)$$
$$-\cos\vartheta\,\sin2\alpha\,\cos\omega t\,\sin At. \qquad (9)$$

In the ideal Foucault case, $\alpha=\pi/4$, this reduces to

$$\psi_F(t)=x(t)=\cos\omega t\,\cos(\vartheta-\Omega_z t)$$
$$=y(t)=\cos\omega t\,\sin(\vartheta-\Omega_z t). \qquad (10)$$

The orbit is the well-known linear oscillation whose direction tends to move in the same direction to a terrestrial observer as the sun. For the linear eigenfunctions found on the equator, $\alpha=0$, Eq. (9) reduces to

$$\psi_L(t)=x(t)=\cos\vartheta\,\cos[(\omega+\delta)t]$$
$$=y(t)=\sin\vartheta\,\cos[(\omega-\delta)t]. \qquad (11)$$

These orbits are ellipses bounded by the rectangle $x=\pm\cos\vartheta$, $y=\pm\sin\vartheta$, and they include the diagonals as limiting cases. They are easily

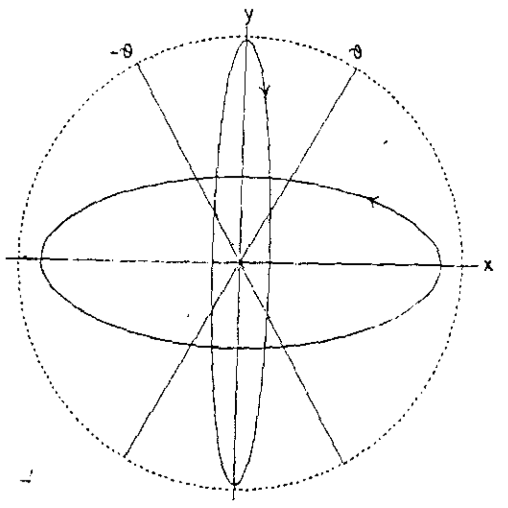

FIG. 1. Characteristic orbits of the Foucault pendulum as discussed in the text. Figures 1–5 are reproduced from Kamerlingh Onnes' thesis, the only modification being the use of the labeling of coordinates in Figs. 1–3.

(1) in $\delta=0$ $\cos\alpha=\frac{\sqrt{A+\delta}}{\sqrt{2A}}=\frac{\sqrt{A}}{\sqrt{2A}}=\frac{1}{\sqrt{2}}\Rightarrow\alpha=\pi/4$

simulated as Lissajous figures on an oscilloscope. One uses constant voltages of the same frequency, but variable phase difference, for x and y deflection.

An examination of the general pendulum orbit (9) shows the following typical features:

(a) A linear oscillation in the direction ϑ occurs at times t where $At=0, \pi, 2\pi\cdots$.

(b) The orbit is an ellipse with principal axes in the x and y directions at times t given by $At=\tau/2, \pi+\tau/2, 2\pi+\tau/2\cdots$, where τ is the root of the transcendental equation

$$\tan\tau=\tan 2\vartheta/\sin 2\alpha. \tag{12}$$

Looking down to the orbit, the pendulum motion is counterclockwise if $0<\vartheta<\pi/2$.

(c) Another linear oscillation, this one in the direction $-\vartheta$, occurs when $At=\tau, \pi+\tau, 2\pi+\tau\cdots$.

(d) The orbit becomes again an ellipse with principal axes in the x and y directions whenever $At=\pi-\tau/2, 2\pi-\tau/2\cdots$. The pendulum motion is clockwise for the same choice of ϑ.

This behavior is illustrated by Kamerlingh Onnes' drawing, Fig. 1. If the asymmetry of the pendulum is small, i.e., $\delta\ll\Omega_z$, then the motion implies a continuous (although not quite uniform) clockwise rotation of the major axis. A computed example is shown in Fig. 2. For stronger asymmetry the major axis oscillates back and forth as in the example of Fig. 3. During a "normale onderperiode" as Onnes called it, that is for times t when $\tau+n\pi<At<(n+1)\pi$ in the present notation, the major axis moves monotonically from $-\vartheta$ to $+\vartheta$. Thereafter, during an "abnormale

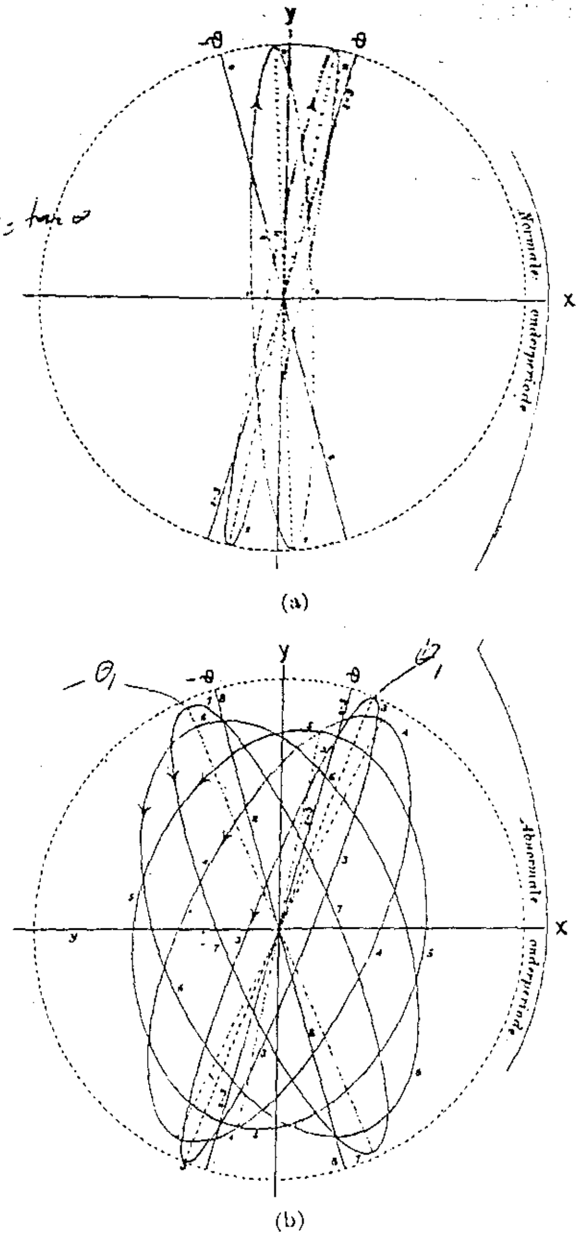

FIG. 3. Series of computed pendulum orbits. During the "normale onderperiode" the major axis moves clockwise, during part of the "abnormale onderperiode" it moves counterclockwise.

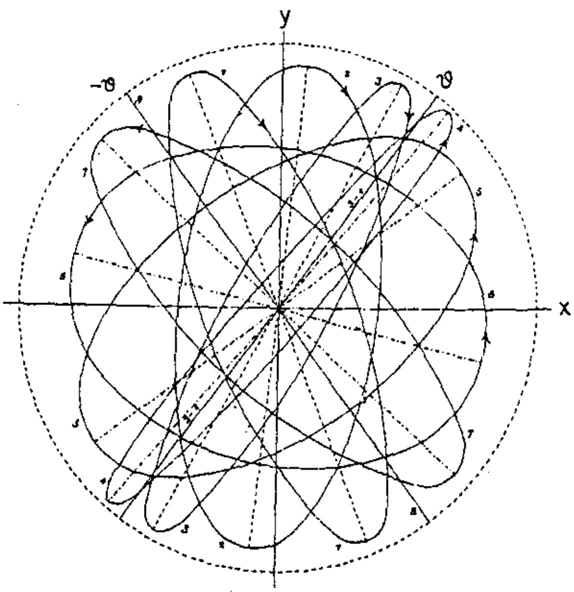

FIG. 2. Series of computed pendulum orbits for the case of "normal" motion, i.e., the major axis of the ellipses moves continually clockwise with time.

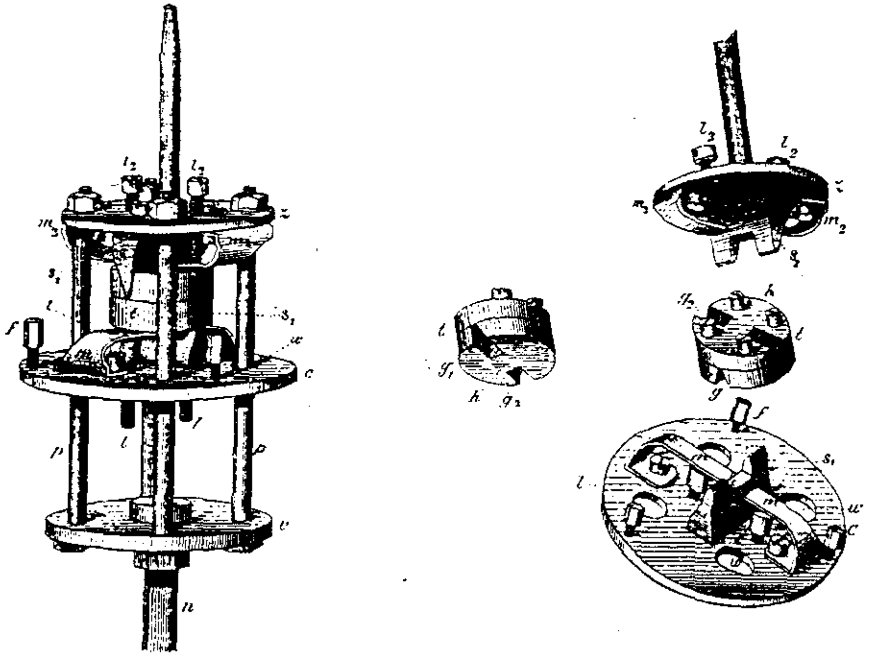

Fig. 4. Assembled and exploded view of the double knife edge suspension. The middle plate is anchored to the tripod while the top and bottom central rods move with the body of the pendulum.

onderperiode," $n\pi < At < n\pi + \tau$, the major axis first moves a little clockwise to $\vartheta_1 > \vartheta$, then reverses its motion and proceeds counterclockwise to $-\vartheta_1$ from where it again moves clockwise to $-\vartheta$.

Earlier investigators did not appreciate the effects of mechanical asymmetry of the pendulum. Thus they were puzzled when they observed elliptical orbits, especially if the time evolution was as complicated as that illustrated in Fig. 3. By the foregoing analysis, on the other hand, Kamerlingh Onnes not only explained the tendency towards elliptical orbits, but he was able to set up a procedure for eliminating the mechanical asymmetry and thus to realize a pendulum with the "ideal Foucault performance." One such procedure is sketched below.

Excite the pendulum to linear oscillations in some arbitrary direction. In the analysis this direction is ϑ. By observing the developing elliptical orbits, find the other direction where the orbit again becomes a linear oscillation. This direction is $-\vartheta$. The angle bisector then defines a principal axis of the mechanical potential, i.e., it would be the x or y direction in the present notation. Now excite the pendulum to linear oscillations in one of these principal directions. After the time t given by $At = \pi/2$, the orbit is an ellipse with principal axes in the x and y directions. The peak excursion in the direction of the excitation is proportional to $\cos 2\alpha$, and that perpendicular to it, proportional to $\sin 2\alpha$. If the excitation is along the x axis, i.e., the direction corresponding to the higher eigenfrequency, then the pendulum moves clockwise on its elliptical orbit. The reverse is true for excitation along the y direction. Observation of major and minor axes of the ellipse and of the sense of circulation thus permits one to find the asymmetry parameter α. This then can be used to compute the mass of a trimming attachment which would eliminate the asymmetry part δ of the frequency splitting. In practice this procedure may have to be repeated.

These comments may suffice to illustrate the significance of degenerate perturbation theory for a quantitative Foucault experiment. Before leaving the subject, however, it may be worthwhile showing some of the equipment which Kamerlingh Onnes designed for his experiment. Figure 4 illustrates the double knife edge suspension of the pendulum. Assuming mechanical perfection, the design guarantees oscillation in any direction about the same origin. The adjustable springs delimit the oscillation amplitude. This

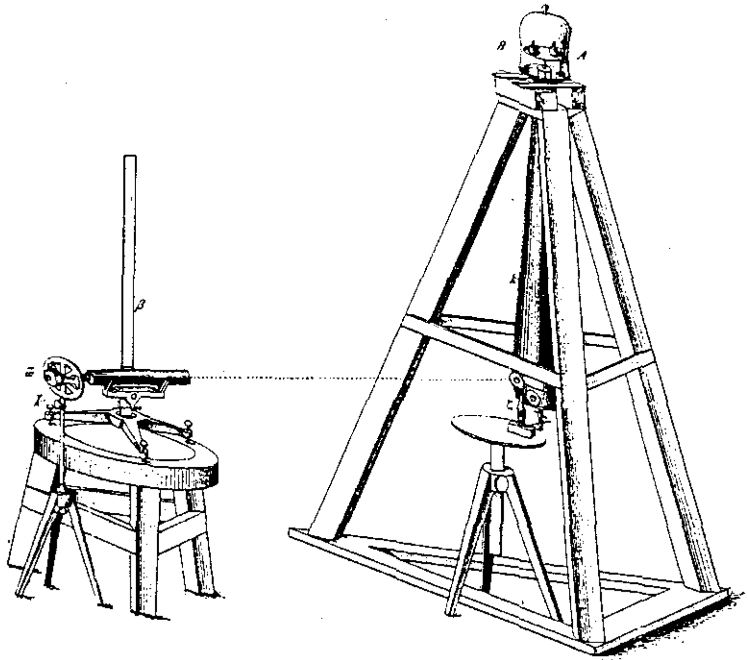

FIG. 5. Sketch of the entire apparatus.

design probably offers more reproducible performance than suspension by a thin wire. Figure 5 shows the assembled apparatus. The pendulum is enclosed in a gas-tight chamber to eliminate drafts. A small platform is attached on top of the shaft of the pendulum. It is visible under the glass cover at the top of the tripod. Weights placed on the platform are used to correct for the unavoidable mechanical asymmetry of the pendulum. A fairly sophisticated optical system is used to monitor the pendulum motion. It consists of a light source, two deflection prisms near the pendulum bob, telescope and goniometer with crosshairs. Attached to the pendulum bob is an iris whose motion in and out of the optical path can be observed. It is probably fair to say that the careful design of this apparatus is an early testimony to the experimental skills which later won world fame for Kamerlingh Onnes.

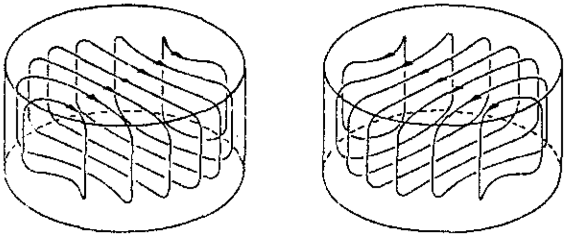

FIG. 6. The two orthogonal degenerate TE_{111} modes in a circular cylindrical cavity. This and the following sketch show only the rf magnetic field.

DEGENERATE PERTURBATION THEORY OF MICROWAVE CAVITIES

There are experimental techniques which require the realization of microwave cavities with circularly polarized magnetic (or electric) rf field. A commonly used cavity geometry is the circular cylinder. As illustrated in Fig. 6, two degenerate TE_{111} modes can exist there. If they are excited with equal amplitude and in phase quadrature, the resulting rf magnetic (and electric) field near the axis is circularly polarized. Another suitable geometry is the parallelepiped with square cross-section where the TE_{101} and TE_{110} modes are degenerate. Figure 7 shows a rectangular cavity which is essentially a doubled square cross-section cavity. Here the quadrature excitation produces two regions in which there is circular polarization, however, of the opposite sense.

Such cavities are used, for example, to study gyromagnetic materials whose permeability in a